Identical particles and pauli's principle:-

Identical particles:- exchange symmetry of wave functions

- By identical particles we mean that if the position and spin coordinates of any two of them are interchanged, there is no physical way of knowledge that a particle change has been made in the system.

Suppose a box contains two identical particles say electrons. classically the electrons travel in sharply defined trajectories so that the observing the system constantly we can distinguish b/w them. That is if at some instant we label the electrons as 1 and 2 we can constantly watch their motion without disturbing the system. That is if at some instant w electron is 1 and which is 2.

It is not possiable in quantum mechanical system. The uncertainty principle does not allows us to observe constantly the motion of the electrons without disturbing the system.

In quantum mechanics the finite extent of the wave function associated with each electrons results in an overlapping of these wave functions and it becomes impossiable to say wave function associated with which particle.

Therefore a quantum mechanical description of a system of identical particles should take into account the indistinguishability of the identical particles.

let two e in quantum mechanically,

then $\hat{H} = \hat{H}_1 + \hat{H}_2$ {Hamiltonian}

$\Psi_{(1,2)} = \psi(1)\,\psi(2)$ $\begin{cases} 1 \rightarrow \text{Quantum state } a \\ 2 \rightarrow \quad a \quad b \end{cases}$

$\psi(1,2) = \psi_a(1)\,\psi_b(2)$ —— (1)

$\psi_{ab} = \psi_a(1)\,\psi_b(2)$

A wave function indicating that electron 2 is in state b is

$\psi_{ba} = \psi_b(1)\,\psi_a(2)$ — (i)

prob. density function of the system (1 & 2)

$\psi^*_{ab}\,\psi_{ab} = \psi^*_a(1)\,\psi^*_b(2)\,\psi_a(1)\,\psi_b(2)$ — (iii)

$\psi^*_{ba}\,\psi_{ba} = \psi^*_b(1)\,\psi^*_a(2)\,\psi_b(1)\,\psi_a(2)$ — (iv)

So, $\psi_a^*(1)\,\psi_b^*(2)\,\psi_a(1)\,\psi_b(2) \longrightarrow \psi_a^*(2)\,\psi_b^*(1)\,\psi_a(2)\,\psi_b(1)$

$1 \to 2$

$2 \to 1$

Now, $\psi(x_1, y_1, z_1, s_{z1}, x_2, y_2, z_2, s_{z2} \ldots x_N, y_N, z_N)$

$\psi(1, 2, 3, \ldots N)$

The Hamiltonian operator is

$$\hat{H} = -\frac{h^2}{8\pi^2 m}\left(\nabla_1^2 + \nabla_2^2 + \nabla_3^2 + \ldots \nabla_N^2\right) + V(1,2,3,\ldots N)$$

If any two particles say 1 and 2 are interchanged the Hamiltonian becomes

$$\hat{H}' = -\frac{h^2}{8\pi^2 m}\left(\nabla_2^2 + \nabla_1^2 + \nabla_3^2 + \ldots \nabla_N^2\right) + V(2,1,3,\ldots N)$$

Since the potential energy must be the same to the positions and spins of the particles.

i.e. $\boxed{\hat{H}' = \hat{H}}$ → exchange variation of Hamiltonian

$$\hat{P}_{12}\,\psi(1,2,3,\ldots N) = \psi(2,1,3,\ldots N) \quad \text{— (i)}$$

parity operator $\hat{P}_{12}$ is also a linear, Hermitian operator having eigen is ± 1 only

$$\hat{P}_{12}\psi = \alpha\psi \quad \text{— (ii)}$$

where α is the eigen value.

operating once again

$$\hat{P}^2_{12}\psi = \alpha^2\psi$$

$$\hat{P}^2_{12}\psi = \psi$$

$\therefore \alpha^2 = 1 \;\Rightarrow\; \boxed{\alpha = \pm 1}$

$\hat{P}_{12}\psi(1,2,3,\cdots N) = \pm \psi(1,2,3,\cdots N)$.

Comparing eqn (1)

$$\boxed{\psi(2,1,3,\cdots N) = \pm \psi(1,2,3,\cdots N)}$$

(1) If $\boxed{\psi(2,1,3\cdots N) = +\psi(1,2,3\cdots N)}$,

Those particles which can be described only by symmetric wave functions are called Bose particles or Bosons. Its spin is integral

Photon → spin 1

Deuteron → 1

α particles → 0

Ground He atom → 0

(2) Those particles which can be described only by anti-symmetric total wave functions are called Fermi-particles or Fermions.

$$\boxed{\psi(1,2,3,\cdots N) = -\psi(1,2,3,\cdots N)}$$

Spin → half integral.

electrons, protons, neutron, μ-mesons, are Fermions.

Formulation of Pauli's principle:

1925 discovered → No two e^- in an atom can exist in the same quantum state.

$$H^1 = \left\{-\frac{h^2}{8\pi^2 m}\nabla_1^2 + V(x_1, y_1, z_1, s_{z_1})\right\} + \left\{-\frac{h^2}{8\pi^2 m}\nabla_2^2 + V(x_2, y_2, z_2, s_{z_2})\right\}$$

i.e $H^1 = H_1^1 + H_2^1$

$\psi(1,2) = \psi_a(1)\ \psi_b(2)$

$\psi_{as}(1,2) = \psi_a(1)\ \psi_b(2)$

$\psi_{sa}(1,2) = \psi_b(1)\ \psi_a(2)$

Th. $\psi(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) \pm \psi_b(1)\psi_a(2)\right]$

The factor $\frac{1}{\sqrt{2}}$ is normalize to $\psi(1,2)$.

Eg, $\psi_{Bose}(1,2) = \frac{1}{\sqrt{2}}[\psi_a(1)\psi_b(2) + \psi_b(1)\psi_a(2)]$ symmetric

$\psi_{Fermi}(1,2) = \frac{1}{\sqrt{2}}[\psi_a(1)\psi_b(2) - \psi_b(1)\psi_a(2)]$

If $a = b$

antisymmetric

$\psi_{Bose}(1,2) \neq 0$

$\psi_{Fermi} = 0$

Slater determinant :-

An antisymmetric wave function for a system of two non-interacting (Fermi) particles can be written as a determinant

$$\psi_{Fermi}(1,2) = \frac{1}{\sqrt{2}}[\psi_a(1)\psi_b(2) - \psi_b(1)\psi_a(2)]$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_a(1) & \psi_a(2) \\ \psi_b(1) & \psi_b(2) \end{vmatrix}$$

Therefore, a system of N non-interacting Fermi particles written as,

$$\psi_{Fermi}(1,2,\ldots N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_a(1) & \psi_b(2) & \cdots & \psi_n(N) \\ \psi_b(1) & \psi_b(2) & \cdots & \psi_b(N) \\ \vdots & \vdots & & \vdots \\ \psi_n(1) & \psi_n(2) & \cdots & \psi_n(N) \end{vmatrix}$$

This is called Slater determinant.

Since the value of this determinant consists of sum and difference of N! terms comprised of products of one-particle wavefunction